Calculus 1 IB – Logarithm Bases – SL Type I

Section 1

Four given sequences are below and two extra terms in each sequence are given in bold.

$\log_2 8, \log_4 8, \log_8 8, \log_{16} 8, \log_{32} 8, \boldsymbol{\log_{64} 8, \log_{128} 8}$

The bases of the logs in this sequence doubled each time starting from 2.

$\log_3 81, \log_9 81, \log_{27} 81, \log_{81} 81, \boldsymbol{\log_{243} 81, \log_{729} 81}$

The bases of the logs in this sequence are tripled each time starting from 3.

$\log_5 25, \log_{25} 25, \log_{125} 25, \log_{625} 25, \boldsymbol{\log_{3125} 25, \log_{15625} 25}$

The bases of the logs in this sequence are multiplied by 5 starting from 5.

$\log_{m^1} m^k, \log_{m^2} m^k, \log_{m^3} m^k, \log_{m^4} m^k, \boldsymbol{\log_{m^5} m^k, \log_{m^6} m^k}$

The bases of the logs in this sequence are raised to n starting where $n = 1$.

Section 2

In terms of p and q where $p, q \in \mathbb{Z}$, expressions for the $\mathrm{n_{th}}$ term for each sequence above can be written in the form of $\frac{p}{q}$. For any sequence, p is the answer of the first term and q is n.

$\log_2 8 = 3 = p$ and $q = n$, therefore $\boldsymbol{\frac{3}{n}}$

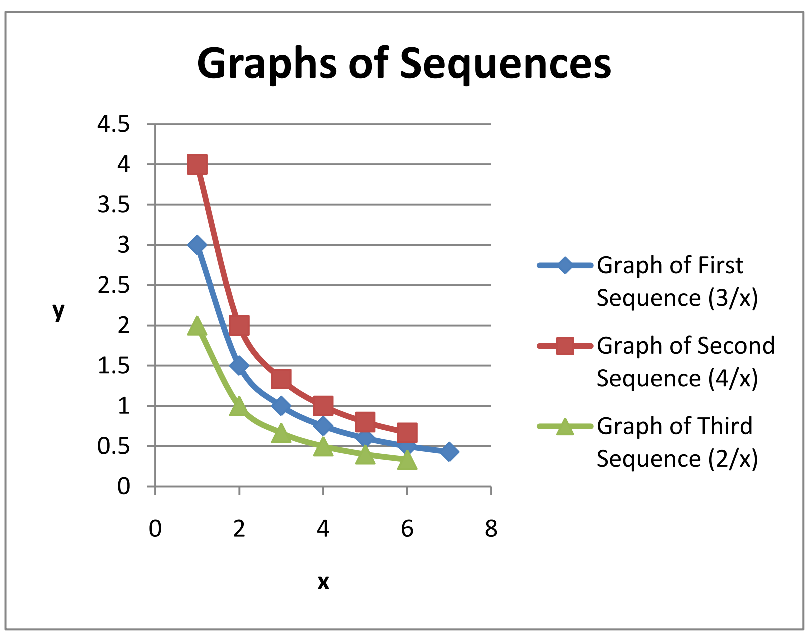
$\log_3 81 = 4 = p$ and $q = n$, therefore $\boldsymbol{\frac{4}{n}}$

$\log_5 25 = 2 = p$ and $q = n$, therefore $\boldsymbol{\frac{2}{n}}$

$\log_{m^1} m^k = k = p$ and $q = n$, therefore $\boldsymbol{\frac{k}{n}}$

The graph to the right represents a *power regression*, via a calculator, applied to the answers of the first three sequences above along with the respective point positions for each answer in each sequence.

The point positions shown fall on the regression curves. The expressions that can find the $\mathrm{n_{th}}$ term above also match the regression lines.



Section 3

The following can be answered in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$. These logarithms were solved via a calculator.

$\log_4 64 = \frac{\mathbf{3}}{\mathbf{1}}, \log_8 64 = \frac{\mathbf{2}}{\mathbf{1}}, \log_{32} 64 = \frac{\mathbf{6}}{\mathbf{5}}$

$\log_2 49 = \frac{\mathbf{2}}{\mathbf{1}}, \log_{49} 49 = \frac{\mathbf{1}}{\mathbf{1}}, \log_{343} 49 = \frac{\mathbf{2}}{\mathbf{3}}$

$\log_{\frac{1}{5}} 125 = -\frac{\mathbf{3}}{\mathbf{1}}, \log_{\frac{1}{125}} 125 = -\frac{\mathbf{1}}{\mathbf{1}}, \log_{\frac{1}{625}} 125 = -\frac{\mathbf{3}}{\mathbf{4}}$

$\log_8 512 = \frac{\mathbf{3}}{\mathbf{1}}, \log_2 512 = \frac{\mathbf{9}}{\mathbf{1}}, \log_{16} 512 = \frac{27}{12} = \frac{\mathbf{9}}{\mathbf{4}}$

Section 4

The third answer in each row can be found by using the first two answers in each row. To find the third answer from each row by using the first two, divide the product of the first two answers by the sum of the first two answers. These logarithms were via this method and checked via a calculator.

An example using the formula first row's answers, $\frac{3}{1}, \frac{2}{1}$ and $\frac{6}{5}$: $\frac{3*2}{2+3} = \frac{6}{5}$

Two more examples that fit the pattern above:

$\log_4 256 = \frac{4}{1}, \log_{16} 256 = \frac{2}{1}, \log_{64} 256 = \frac{8}{6} = \frac{\mathbf{4}}{\mathbf{3}}$

$a = \frac{4}{1}, b = \frac{2}{1}, c = \frac{4*2}{4+2} = \frac{8}{6} = \frac{\mathbf{4}}{\mathbf{3}}$

$\log_2 1024 = \frac{10}{1}, \log_4 1024 = \frac{5}{1}, \log_8 1024 = \frac{50}{15} = \frac{\mathbf{10}}{\mathbf{3}}$

$a = \frac{10}{1}, b = \frac{5}{1}, c = \frac{10*5}{10+5} = \frac{50}{15} = \frac{\mathbf{10}}{\mathbf{3}}$

To make the pattern more apparent, this is a table contains the results of each row above, including the additional examples, in unreduced form.

Table of Answers (From Section 3 and 4)						
First Answer	$\frac{3}{1}$	$\frac{2}{1}$	$-\frac{3}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{10}{1}$
Second Answer	$\frac{2}{1}$	$\frac{1}{1}$	$-\frac{1}{1}$	$\frac{9}{1}$	$\frac{2}{1}$	$\frac{5}{1}$
Third Answer	$\frac{6}{5}$	$\frac{2}{3}$	$-\frac{3}{4}$	$\frac{27}{12}$	$\frac{4}{3}$	$\frac{50}{15}$

<u>Section 5</u>

Given $\log_a x = c$ and $\log_b x = d$, $\log_{ab} x$ can be *generalized* as $\frac{\log_a x \log_b x}{\log_a x + \log_b x}$ or in terms of c and d, $\boldsymbol{\frac{cd}{c+d}}$.

The following examples made use of a calculator to do approximate calculations and where necessary.

Example 1

Set a, b, and x to 3, 9 and 2187 respectively.

Let $a = 3, b = 9, x = 2187$

$\log_{ab} x$ becomes $\log_{27} 2187$ with these values.

$$\log_{ab} x = \log_{3*9} 2187 = \log_{27} 2187$$

$\log_a x = c$ and $\log_b x = d$

$\log_3 2187 = c$ and $\log_9 2187 = d$.

Letting $\log_a x = c$ and $\log_b x = d$, c and d are equal to 7 and 3.5 respectively.

$$d = 3.5 = \frac{7}{2}, c = 7 = \frac{7}{1}$$

Using the general statement, value of $\log_{27} 2187$ can be found and subsequently tested by solving $\log_{27} 2187$ on a calculator.

$$\frac{\left(\frac{7}{1}\right)\left(\frac{7}{2}\right)}{\frac{7}{1} + \frac{7}{2}} = \frac{24.5}{10.5} = \frac{7}{3}$$

$$\log_{27} 2187 = \frac{7}{3}$$

Example 2

Set a, b, and x to .5, .7 and 5 respectively.

Let $a = .5, b = .7, x = 5$

$\log_{ab} x$ becomes $\log_{.35} 5$ with these values.

$$\log_{ab} x = \log_{.5*.7} 5 = \log_{.35} 5$$

$\log_a x = c$ and $\log_b x = d$

$\log_{.5} 5 = c$ and $\log_{.7} 5 = d$.

Letting $\log_a x = c$ and $\log_b x = d$, c and d are approximately equal to -2.32 and -4.51 respectively.

$$d = -4.51, c = -2.32$$

Using the general statement, value of $\log_{.35} 5$ can be found and subsequently tested by solving $\log_{.35} 5$ on a calculator.

$$\frac{(-2.32)(-4.51)}{-2.32 + -4.51} \approx \frac{10.5}{-6.83} \approx -1.54$$

$$\log_{.35} 5 \approx -1.53$$

Example 3

Set a, b, and x to 3, 7 and 139 respectively.

Let $a = 3, b = 7, x = 139$

$\log_{ab} x$ becomes $\log_{21} 139$ with these values.

$$\log_{ab} x = \log_{3*7} 139 = \log_{21} 139$$

$\log_a x = c$ and $\log_b x = d$

$\log_3 139 = c$ and $\log_7 139 = d$.

Letting $\log_a x = c$ and $\log_b x = d$, c and d are approximately equal to 4.49 and 2.54 respectively.

$$d \approx 2.54, c \approx 4.49$$

Using the general statement, value of $\log_{21} 139$ can be found and subsequently tested by solving $\log_{21} 139$ on a calculator.

$$\frac{(4.49)(2.54)}{4.49 + 2.54} \approx \frac{11.4}{7.03} \approx 1.62$$

$$\log_{21} 139 \approx 1.62$$

Scope and Limitations

Because logarithms cannot have a negative base, have a negative x value, equal 0 or 1, the limitations are:

$$a > 0, a \neq 1$$

$$b > 0, b \neq 1$$

$$x > 0, x \neq 1$$

Outside of these constraints, all values are valid.

Section 6

The general statement, given $\log_a x = c$ and $\log_b x = d$, $\log_{ab} x$ can be generalized as $\frac{\log_a x \log_b x}{\log_a x + \log_b x}$ or in terms of c and d, $\frac{cd}{c+d}$, was found by finding a connection with the first two answers and the third answer in section 3 and 4 above. It was noticed that the first two answers, when multiplied equals the numerator of the third answer and when summed equals the denominator of the third answer. This ended up being the expanded form, $\frac{\log_a x \log_b x}{\log_a x + \log_b x}$ but this simplified into $\frac{cd}{c+d}$.

The general statement be justified via symbolic algebra. This justification assumes that a, b and x are within the scope and limitations, mentioned above.

Given $\log_a x = c$ and $\log_b x = d$, $\log_{ab} x$ can be generalized as $\frac{\log_a x \log_b x}{\log_a x + \log_b x}$ or in terms of c and d, $\frac{cd}{c+d}$.

$$\log_a x =, \log_b x = d$$

$$a^c = x, b^d = x$$

$$a = x^{\frac{1}{c}}, b = x^{\frac{1}{d}}$$

$$ab = \left(x^{\frac{1}{c}}\right)\left(x^{\frac{1}{d}}\right) = x^{\frac{1}{c}+\frac{1}{d}}$$

$$x = ab^{\frac{1}{\frac{1}{c}+\frac{1}{d}}}$$

$$x = ab^{\frac{cd}{c+d}}$$

$$\log_{ab} x = \frac{cd}{c+d}$$